

# ARTIN REPRESENTATIONS FOR $GL_n$

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**ABSTRACT.** Let  $\pi$  be a cuspidal automorphic representation of  $GL_n(\mathbb{A}_{\mathbb{Q}})$  which satisfies certain reasonable assumptions such as integrality of Hecke polynomials, the existence of mod  $\ell$  Galois representations attached to  $\pi$ . Under Langlands functoriality of exterior  $m$ -th power  $\wedge^m(\pi)$ ,  $m = 2, \dots, [\frac{n}{2}]$ , we will construct a unique Artin representation associated to  $\pi$ . As a corollary, we obtain that such a cuspidal representation of  $GL_n(\mathbb{A})$  satisfies the Ramanujan conjecture. We also revisit our previous work on Artin representations associated to non-holomorphic Siegel cusp forms of weight  $(2,1)$ , and show that we can associate non-holomorphic Siegel modular forms of weight  $(2,1)$  to Maass forms for  $GL_2/\mathbb{Q}$  and cuspidal representations of  $GL_2$  over imaginary quadratic fields.

## 1. INTRODUCTION

The purpose of this paper is, under certain reasonable assumptions, to associate an irreducible complex Galois representation, called Artin representation,  $\rho : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$ , to a cuspidal representation  $\pi$  of  $GL_n(\mathbb{A}_{\mathbb{Q}})$  which satisfies certain properties. See Section 2 for the notations. The assumptions are:

- (1) Langlands functoriality of the exterior  $m$ -th power  $\wedge^m(\pi)$  as an automorphic representation of  $GL_{C_{n,m}}(\mathbb{A}_{\mathbb{Q}})$ , where  $C_{n,m} = \binom{n}{m} := \frac{n!}{m!(n-m)!}$ ,  $m = 2, \dots, [\frac{n}{2}]$ ;
- (2) the existence of mod  $\ell$  Galois representation attached to  $\pi$ ;
- (3) finiteness of Hecke field of  $\pi$ , i.e., rationality of Satake parameters;
- (4) the integrality of Hecke polynomials of  $\pi$ ;
- (5) the existence of Galois conjugates of  $\pi$ .

Under these assumptions, we will prove the following main theorem:

**Theorem 1.1.** (*Main Theorem*) *Let  $\pi$  satisfy the above assumptions. Let  $N$  be the conductor of  $\pi$  so that  $\pi_p$  is unramified for  $p \nmid N$ . Then there exists the Artin representation  $\rho_{\pi} : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$*

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which is unramified for  $p \nmid N$  such that

$$\det(1_n - \rho_F(\text{Frob}_p)T) = H_p(T)$$

for all  $p \nmid N$ . Furthermore,  $\rho_\pi(c) \stackrel{GL_n(\mathbb{C})}{\sim} \text{diag}(\epsilon_1, \dots, \epsilon_n)$ ,  $\epsilon_i \in \{\pm 1\}$  for the complex conjugate  $c$ , and  $\pi_\infty \simeq \pi(\epsilon'_1, \dots, \epsilon'_n)$  where  $\epsilon'_i = \begin{cases} 1, & \text{if } \epsilon_i = 1 \\ \text{sgn}, & \text{if } \epsilon_i = -1 \end{cases}$ .

As a corollary, we obtain that the above assumptions on  $\pi$  imply the Ramanujan conjecture for  $\pi$ , namely,  $\pi_p$  is tempered for all  $p$ .

Theorem 1.1 is a generalization of the result of Deligne and Serre who associated an odd irreducible Artin representation  $\rho_f : G_{\mathbb{Q}} \rightarrow GL_2(\mathbb{C})$  to any elliptic cusp form  $f$  of weight one [8]. The assumptions force  $\pi_\infty$  to be a principal series representation of the form  $\pi(\epsilon'_1, \dots, \epsilon'_n)$ . Naively we hope that this kind of automorphic representation  $\pi$  should satisfy the above assumptions (2)-(5). However, unlike holomorphic modular forms in  $GL_2/\mathbb{Q}$  case where one can use algebraic geometry as Deligne and Serre did, it seems difficult in general  $GL_n$  case to verify whether a given  $\pi$  satisfies these strong assumptions. Note that in holomorphic modular forms in  $GL_2$  case,  $\pi_\infty$  is a limit of discrete series. But in the case of  $GL_n$ ,  $n \geq 3$ ,  $\pi_\infty$  is not a limit of discrete series (cf. [17]).

A main innovation in this paper is to prove some structure theorem (see Section 5) for semisimple finite subgroups of linear algebraic groups over finite fields by using result of Larsen-Pink [18] with the appendix in [13]. This enables us to generalize Proposition 7.2 of [8] to general linear groups. As in [8] and [16] (see also [25] in the case of totally real fields), it is done by a brute force with the complete classification of the finite semisimple subgroups in  $GL_n(\mathbb{F}_p)$  of small degrees. In general, it seems to be difficult to list all of such groups even if  $n$  is fixed. Therefore some sophisticated analysis is required so that we do not need explicit forms of subgroups in question at hand. We remark that this is not at all obvious from the existing results in the literature (cf. [18] and [22]).

The organization of this paper is as follows. In Section 2, we state precisely the assumptions which we imposed on  $\pi$ . In Section 3, we investigate the infinity type of an automorphic representation of  $GL_n(\mathbb{A}_{\mathbb{Q}})$  which gives rise to an Artin representation. In Section 4, we apply Rankin-Selberg method to prove that the number of Satake parameters for  $\pi_p$  outside a certain infinite set of primes is finite. As mentioned above, in Section 5, we study finite subgroup of linear groups over a finite field. Then we apply results in Section 5 with the Rankin-Selberg method

to prove the boundedness of the size of the image of mod  $\ell$  Galois representations provided that they exist. The formulation of the conjecture for such mod  $\ell$  Galois representations is given in Section 6. The proof of the main theorem is given in Section 7 along the method of Deligne-Serre.

In Section 8 and Section 9, we recall our previous work [16] for the case of  $GSp_4/\mathbb{Q}$ . We discuss a relation between non-holomorphic Siegel modular forms and holomorphic Siegel modular forms. After the completion of the previous work, we realized that we do not need to use the unproved hypothesis on the existence of the weak transfer from  $GSp_4$  to  $GL_4$  as in [2]. We explain how to get around this. Finally in Section 10, we consider Maass forms for  $GL_2/\mathbb{Q}$  and automorphic representations of  $GL_2$  over imaginary quadratic fields. We show that we can associate non-holomorphic Siegel modular forms of weight  $(2, 1)$  to them.

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## 2. ASSUMPTIONS ON CUSPIDAL REPRESENTATIONS

Let  $\pi = \pi_\infty \otimes \otimes'_p \pi_p$  be a cuspidal representation of  $GL_n/\mathbb{Q}$ . Let  $N$  be the conductor of  $\pi$  so that  $\pi_p$  is unramified for  $p \nmid N$ . Let  $\{\alpha_1(p), \dots, \alpha_n(p)\}$  be the Satake parameters for  $p \nmid N$ . Let  $\omega$  be the central character of  $\pi$ . Then  $\omega(p) = \alpha_1(p) \cdots \alpha_n(p)$ .

For  $1 \leq m \leq n$ , let  $\wedge^m : GL_m(\mathbb{C}) \rightarrow GL_{C_{n,m}}(\mathbb{C})$  be the exterior  $m$ -th power. By the local Langlands correspondence, one can define  $\Pi_m = \wedge^m(\pi)$  as an admissible representation of  $GL_{C_{n,m}}(\mathbb{A}_{\mathbb{Q}})$ . Langlands functoriality conjecture says that the exterior  $m$ -th power  $\Pi_m = \wedge^m(\pi)$  is an automorphic representation of  $GL_{C_{n,m}}/\mathbb{Q}$ . We assume this in this paper. Since  $\wedge^m(\pi) = \wedge^{n-m} \tilde{\pi} \otimes \omega_\pi$ , it is enough to assume it for  $m \leq \lfloor \frac{n}{2} \rfloor$ . Note that this is known for  $n \leq 4$  [15].

Let

$$L(s, \wedge^m(\pi)) = L(s, \pi, \wedge^m) = \sum_{n=1}^{\infty} a_m(n) n^{-s}.$$

be the exterior  $m$ -th power  $L$ -function, where

$$a_1(p) = \alpha_1(p) + \dots + \alpha_n(p), \quad a_2(p) = \sum_{i < j} \alpha_i(p) \alpha_j(p), \quad \dots, \quad a_n(p) = \omega(p).$$

Consider the Rankin-Selberg  $L$ -function  $L(s, \Pi_m \times \tilde{\Pi}_m)$ . Write

$$L(s, \Pi_m \times \tilde{\Pi}_m) = \sum_{n=1}^{\infty} \frac{b_m(n)}{n^s},$$

where  $b_m(p) = |a_m(p)|^2$ . Note that  $L(s, \Pi_m \times \tilde{\Pi}_m)$  has a pole at  $s = 1$ , of order at least 1, and at most  $C_{n,m}^2$ . Since  $L(s, \Pi_m \times \tilde{\Pi}_m) = \sum_p \frac{|a_m(p)|^2}{p^s} + g(s)$  for a holomorphic function  $g(s)$  near  $s = 1$ ,

$$\sum_p \frac{|a_m(p)|^2}{p^s} \leq C_{n,m}^2 \log \frac{1}{s-1} + O(1), \quad \text{as } s \rightarrow 1^+.$$

Let  $\mathbb{Q}_\pi = \mathbb{Q}(a_m(p), m = 1, \dots, n, p \nmid N)$  be the Hecke field of  $\pi$ . Let  $K$  be the Galois closure of  $\mathbb{Q}_\pi$ , and  $\mathcal{O}_K$  be the ring of integers of  $K$ . Since

$$H_p(T) := (1 - \alpha_1(p)T) \cdots (1 - \alpha_n(p)T) = 1 - a_1(p)T + \cdots + (-1)^n a_n(p)T^n,$$

$\mathbb{Q}_\pi = \mathbb{Q}(\alpha_1(p), \dots, \alpha_n(p), p \nmid N)$ . So  $a_1(p), \dots, a_n(p)$  take only finitely many values if and only if  $\alpha_i(p)$ ,  $i = 1, \dots, n$ , takes only finitely many values.

We assume the following:

- (1)  $\mathbb{Q}_\pi$  is a finite extension of  $\mathbb{Q}$ . So  $K$  is a finite extension of  $\mathbb{Q}$ .
- (2)  $a_m(p) \in \mathcal{O}_K[\frac{1}{N}]$  for  $m = 1, \dots, n$  and  $p \nmid N$ .
- (3) For each  $\sigma \in \text{Gal}(K/\mathbb{Q})$ ,  ${}^\sigma \pi$  is a cuspidal representation of  $GL_n/\mathbb{Q}$  with the Satake parameters  $\{{}^\sigma \alpha_1(p), \dots, {}^\sigma \alpha_n(p)\}$ .

### 3. INFINITY TYPE OF ARTIN REPRESENTATION

We show that the cuspidal representation  $\pi$  of  $GL_n/\mathbb{Q}$  which we are considering has a very special infinity type  $\pi_\infty$ . Recall from [21], Appendix,

**Proposition 3.1.** *Let  $\rho : G_{\mathbb{Q}} \rightarrow GL_n(\mathbb{C})$  be an irreducible continuous Galois representation. Let  $\pi$  be a cuspidal automorphic representation of  $GL_n(\mathbb{A}_{\mathbb{Q}})$  such that  $L(s, \pi_p) = L(s, \rho_p)$  for almost all  $p$ . Then  $L(s, \pi_p) = L(s, \rho_p)$  for all  $p$ , and  $L(s, \pi_\infty) = L(s, \rho_\infty)$ .*

Hence if  $\rho : G_{\mathbb{Q}} \rightarrow GL_n(\mathbb{C})$  is associated to  $\pi$ ,  $L(s, \rho_p) = L(s, \pi_p)$  for almost all  $p$ . Then the above proposition says that  $L(s, \rho_\infty) = L(s, \pi_\infty)$ . Hence the Langlands parameter of  $\pi_\infty$  is

$$\phi : W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times \rightarrow GL_n(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \text{diag}(\epsilon_1, \dots, \epsilon_n)$$

where  $\epsilon_i \in \{\pm 1\}$ . Then  $\pi_\infty$  is the full induced representation  $\pi(\epsilon'_1, \dots, \epsilon'_n)$ , where  $\epsilon'_i = \begin{cases} 1, & \text{if } \epsilon_i = 1 \\ \text{sgn}, & \text{if } \epsilon_i = -1 \end{cases}$ .

## 4. APPLICATION OF RANKIN-SELBERG METHOD

Let  $\pi$  be a cuspidal representation of  $GL_n/\mathbb{Q}$  as in Section 2. We prove

**Proposition 4.1.** *For any positive integer  $\eta$ , there exists a set  $X_\eta$  of rational primes such that  $\text{den.sup}X_\eta \leq \eta$ , and the set  $\{(a_1(p), \dots, a_n(p)) \mid p \notin X_\eta\}$  is a finite set, or equivalently,  $\{\text{Satake parameters at } p \mid p \notin X_\eta\}$  is finite.*

Here  $\text{den.sup}X_\eta$  is defined by

$$\limsup_{s \rightarrow 1^+} \frac{\sum_{p \in X_\eta} p^{-s}}{\log \frac{1}{s-1}}.$$

We also define the Dirichlet density  $\text{den}(X_\eta)$  by

$$\lim_{s \rightarrow 1^+} \frac{\sum_{p \in X_\eta} p^{-s}}{\log \frac{1}{s-1}}.$$

For  $c > 0$ , consider two sets:

$$\begin{aligned} Y(c) &= \{a \in \mathcal{O}_K \mid |\sigma(a)|^2 \leq c \text{ for any } \sigma \in \text{Gal}(K/\mathbb{Q})\}, \\ X(c) &= \{p \mid \text{at least one of } a_m(p), m = 1, \dots, n, \text{ does not belong to } Y(c)\}. \end{aligned}$$

Note that since  $\mathcal{O}_K$  is a lattice,  $Y(c)$  is a finite set. By the assumption,  $a_m(p) \in \mathcal{O}_K[\frac{1}{N}]$ . If  $p \notin X(c)$ ,  $Na_m(p) \in Y(N^2c)$  for each  $m = 1, \dots, n$ . Hence the set  $\{(Na_1(p), \dots, Na_n(p)) \mid p \notin X(c)\}$  is finite, and so the set  $\{(a_1(p), \dots, a_n(p)) \mid p \notin X(c)\}$  is finite.

By the assumption, for each  $\sigma \in \text{Gal}(K/\mathbb{Q})$ ,  ${}^\sigma\pi$  is a cuspidal representation of  $GL_n/\mathbb{Q}$  with the Satake parameters  $\{{}^\sigma\alpha_1(p), \dots, {}^\sigma\alpha_n(p)\}$ . Hence for each  $m = 1, \dots, n$ ,

$$\sum_p \frac{|\sigma(a_m(p))|^2}{p^s} \leq C_{n,m}^2 \log \frac{1}{s-1} + O(1), \quad \text{as } s \rightarrow 1^+.$$

Let  $r = [K : \mathbb{Q}]$ . If  $p \in X(c)$ , there exists  $m$  such that  $|\sigma_m(a_m(p))|^2 > c$  for some  $\sigma_m \in \text{Gal}(K/\mathbb{Q})$ . Therefore

$$c \sum_{p \in X(c)} p^{-s} \leq \sum_{m=1}^n \sum_{\sigma} \sum_p \frac{|\sigma(a_m(p))|^2}{p^s} \leq \left( \sum_{m=1}^n C_{n,m}^2 \right) r \log \frac{1}{s-1} + O(1), \quad \text{as } s \rightarrow 1^+.$$

Therefore,  $\text{den.sup}X(c) \leq \frac{r}{c} (\sum_{m=1}^n C_{n,m}^2)$ . Take  $c$  such that  $c \geq \frac{r \sum_{m=1}^n C_{n,m}^2}{\eta}$ .

5. BOUNDS FOR THE ORDERS OF CERTAIN SUBGROUPS OF  $GL_n(\mathbb{F}_q)$ 

Fix a positive integer  $n \geq 1$ . Let  $q$  be a power of a rational prime  $p$  and  $\mathbb{F}_q$  be the finite field with  $q$  elements. Let  $G$  be a subgroup of  $GL_n(\mathbb{F}_q)$ . We say  $G$  is semisimple if the natural action of  $G$  on  $V := \mathbb{F}_q^{\oplus n}$  is semisimple or equivalently  $V$  is a semisimple  $G$ -module. We say  $G$  is an irreducible (resp. absolutely irreducible) subgroup of  $GL_n(\mathbb{F}_q)$  if  $V$  (resp.  $V \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ ) is an irreducible  $G$ -module. For positive constants  $N$  and  $\eta$ , ( $0 < \eta < 1$ ), we introduce the following property  $C(\eta, N)$  for  $G$ :

$C(\eta, N)$  : there exists a subset  $H$  of  $G$  such that  $\begin{cases} (i) (1 - \eta)|G| \leq |H|, \\ (ii) |\{\det(1 - hT) \in \mathbb{F}_q[T] \mid h \in H\}| \leq N. \end{cases}$

**Theorem 5.1.** (*Theorem 0.2 of [18]*) *There exists a constant  $J_1(n)$ , depending only on  $n$  such that any finite subgroup  $G$  of  $GL_n(\mathbb{F}_q)$  possesses normal subgroups  $G \supset G_1 \supset G_2 \supset G_3$  such that*

- (1)  $[G : G_1] \leq J_1(n)$ ;
- (2)  $G_1/G_2$  is a direct product of finite simple groups of Lie type in characteristic  $p$  and the number of direct factors is bounded uniformly in  $n$ ;
- (3)  $G_2/G_3$  is abelian of order not divisible by  $p$ ;
- (4)  $G_3$  is a  $p$ -group.

**Remark 5.2.** *In the above theorem, the boundedness of the number of direct factors is not in the statement of Theorem 0.2 of [18]. However, it is implicit in the proof of the theorem. It is important for our purpose.*

**Theorem 5.3.** (*Chap. V, Section 19, Th. 7, of [30]*) *There exists a constant  $J_2(n)$ , depending only on  $n$  such that any solvable subgroup  $G$  of  $GL_n(\mathbb{F}_q)$  possesses a normal subgroup  $N$  such that*

- (1)  $[G : N] \leq J_2(n)$ ,
- (2)  $N$  is conjugate to a subgroup of the group of upper triangular matrices in  $GL_n(\mathbb{F}_q)$ .

**Corollary 5.4.** *Assume that  $p > J_2(n)$ . Let  $G$  be any semisimple, solvable subgroup of  $GL_n(\mathbb{F}_q)$ . If  $G'$  is a normal  $p$ -subgroup of  $G$ , then  $G'$  is trivial.*

*Proof.* By Theorem 5.3, there exists a normal subgroup  $N$  such that  $[G : N] < p$ . Hence the  $p$ -group  $G'$  is a subgroup of  $N$ . By Theorem 5.3,  $N$  is conjugate to a subgroup of the group of upper triangular matrices and in particular we may assume that  $G'$  is a subgroup consisting of

unipotent, upper triangular matrices. By Clifford's theorem [1], p. 17,  $G'$  is semisimple. Hence  $G' = 1$ .  $\square$

**Corollary 5.5.** *Assume that  $p > J_2(n)$ . Let  $G, G_1, G_2, G_3$  be as in Theorem 5.1. Then  $G_3 = 1$ .*

*Proof.* Since  $G_2$  is solvable, Corollary 5.4 implies the assertion.  $\square$

**Theorem 5.6.** *(cf. Theorem 0.1 of [18]) Let  $k$  be an algebraically closed field of characteristic zero. For every  $n$ , there exists a constant  $J_3(n)$ , depending only on  $n$  such that any finite subgroup  $G$  of  $GL_n(k)$  possesses an abelian normal subgroup  $A$  such that  $[G : A] \leq J_3(n)$ .*

Henceforth we fix  $n \geq 1$  and a positive number  $C(n)$  so that

$$C(n) > \max\{3, n - 3, J_1(n), J_2(n), J_3(n)\}.$$

We will implicitly use the assumption  $C(n) > n - 3$  (resp  $C(n) > 3$ ) in the proof of Proposition 5.10 later to apply a result of [12] (resp. to exclude Suzuki-Ree groups from simple groups appear there).

The following lemma is easy to prove:

**Lemma 5.7.** *Let  $G$  be a subgroup of  $GL_n(\mathbb{F}_q)$ , and  $G'$  a subgroup of  $G$ . Let  $[G : G'] < d$ . Then if  $G$  satisfies  $C(\eta, N)$  for  $(0 < \eta < 1/d)$ , then  $G'$  satisfies  $C(d\eta, N)$ .*

*Proof.* Set  $M = [G : G']$ . Take a subset  $H$  from the first property of  $C(\eta, N)$  for  $G$ . Then one can see that

$$|H| \geq (1 - \eta)|G| = (M - M\eta)|G'| \geq (1 - d\eta)|G'|$$

giving the claim.  $\square$

**Proposition 5.8.** *Let  $G$  be a semisimple subgroup of  $GL_n(\mathbb{F}_q)$  with the order not divisible by  $p$ . If  $G$  satisfies  $C(\eta, N)$  for  $(0 < \eta < \frac{1}{C(n)})$ , then  $|G| \leq B$ , where  $B = B(\eta, N, n)$ .*

*Proof.* By assumption, we may assume that  $G$  is a subgroup of  $GL_n(\mathbb{C})$ . For example, we apply Schur-Zassenhaus' theorem (cf. [9], page 829) to the natural projection  $GL_n(W(\mathbb{F}_q)) \rightarrow GL_n(\mathbb{F}_q)$  where  $W(\mathbb{F}_q)$  the ring of Witt vectors and then get a lift  $G$  to  $GL_n(W(\mathbb{F}_q))$ . Then we have only to compose this with an embedding  $W(\mathbb{F}_q) \hookrightarrow GL_n(\mathbb{C})$ .

By Theorem 5.6 there exists an abelian normal subgroup  $A$  of  $G$  such that  $[G : A] \leq J_3(n)$ . By Lemma 5.7,  $A$  satisfies  $C(C(n)\eta, N)$ . We may assume that  $A$  is a subgroup consisting of diagonal

matrices in  $GL_n(\mathbb{F}_q)$ . Then one has

$$(1 - C(n)\eta)|A| \leq |H| \leq n!N$$

giving a bound of  $|A|$ . Since  $[G : A]$  is bounded, so is  $|G| = [G : A] \times |A|$ .  $\square$

**Proposition 5.9.** *Let  $G$  be an irreducible subgroup of  $GL_n(\mathbb{F}_q)$ . Then the following properties hold:*

- (1) *there exists a finite extension field  $\mathbb{F}_{q^r}$  and an absolutely irreducible subgroup  $G' \subset GL_m(\mathbb{F}_{q^r})$  with  $n = rm$  such that  $G$  is isomorphic to  $G'$ . Furthermore, for any  $g \in G$  and the corresponding  $g' \in G'$  under this isomorphism,*

$$f_g(T) = \prod_{\sigma \in \text{Gal}(\mathbb{F}_{q^r}/\mathbb{F}_q)} f_{g'}(T)^\sigma,$$

*where  $f_g(T), f_{g'}(T)$  stand for characteristic polynomials of  $g, g'$ , resp.*

- (2) *the center of  $G'$  is a cyclic subgroup of  $\mathbb{F}_{q^r}^\times I_m \subset GL_m(\mathbb{F}_{q^r})$ .*

*Proof.* By Schur's lemma, the centralizer  $Z = Z_{M_n(\mathbb{F}_q)}(G)$  is a finite division ring. By Wedderburn's theorem,  $Z$  is a finite field over  $\mathbb{F}_q$ , say  $\mathbb{F}_{q^r}$ , since  $Z$  contains  $\mathbb{F}_q I_n$ . Since  $Z^\times$  acts on  $V := \mathbb{F}_q^{\oplus n}$  faithfully,  $\dim_{\mathbb{F}_q} \mathbb{F}_{q^r} = r$  has to divide  $n$ . Put  $m = \frac{n}{r}$ . We view  $V$  as a  $\mathbb{F}_{q^r}$ -module. Then one has a faithful representation  $G \longrightarrow GL(V) \simeq GL_m(\mathbb{F}_{q^r})$ . To be more precise, if we take a basis  $\{e_1, \dots, e_m\}$  of  $V$  as a  $\mathbb{F}_{q^r}$ -module and a generator  $\alpha \in \mathbb{F}_{q^r}$  over  $\mathbb{F}_q$ , then a basis of  $V$  is given by  $\{\alpha^{p^i} e_j \mid 0 \leq i \leq r-1, 1 \leq j \leq m\}$ . Denote by  $G'$  the image of  $G$  under this representation. Then  $G'$  is absolutely irreducible since  $\mathbb{F}_{q^r} = Z = Z_{M_n(\mathbb{F}_q)}(G) = Z_{M_m(\mathbb{F}_{q^r})}(G')$ . The last claim follows from the direct calculation in this explicit basis.

For the claim (2), let  $g$  be an element in the center  $Z(G')$ . Since  $g$  commutes with the action of  $G'$ , it belongs to  $Z_{M_m(\mathbb{F}_{q^r})}(G')^\times = \mathbb{F}_{q^r}^\times$ .  $\square$

Let  $T$  be the group of all diagonal matrices in  $GL_n(\mathbb{F}_q)$ . Let  $G$  be a semisimple subgroup of  $GL_n(\mathbb{F}_q)$ . Assume  $p > C(n)$ . Fix a series of normal subgroups  $G \supset G_1 \supset G_2 \supset G_3$  in Theorem 5.1 for  $G$ . By Corollary 5.5,  $G_3 = 1$ . Assume  $G_1 \neq G_2$  until the end of the proof of the following proposition.

By Clifford's theorem,  $V = \mathbb{F}_q^{\oplus n}$  is a semisimple  $G_1$ -module. Therefore we have a decomposition  $V = \bigoplus_{1 \leq i \leq m} W_i$  into irreducible components as a  $G_1$ -module, where  $\dim_{\mathbb{F}_q} W_i = n_i r_i$  for each  $i$ . By Proposition 5.9, we may assume that each  $(G_1, W_i)$  is absolutely irreducible module over a



field extension  $\mathbb{F}_{q^{r_i}}$  of  $\mathbb{F}_q$ , and we have a faithful representation  $\pi_i : G_1 \longrightarrow GL(W_i) \simeq GL_{n_i}(\mathbb{F}_{q^{r_i}})$ . We denote by  $G^{(i)}$  the image of  $G_1$ . Then we get an injection

$$G_1 \hookrightarrow \prod_{i=1}^m G^{(i)}$$

which is not necessarily surjective, but we will see this map will be an isomorphism under the natural quotients. Note that clearly  $r_i < n$ .

**Proposition 5.10.** *Under the above setting, the following properties hold:*

- (1) *for each  $i = 1, \dots, m$ , the center  $Z(G^{(i)})$  is a subgroup of  $\mathbb{F}_{q^{r_i}}^\times \text{id}_{W_i}$ , and  $G_2$  is a subgroup of  $T$  so that  $\pi_i(G_2) \subset Z(G^{(i)})$ . Further  $G_2 \hookrightarrow \prod_{1 \leq i \leq m} \pi_i(G_2)$ .*
- (2) *For each  $i = 1, \dots, m$ , there exists a simple and simply connected linear algebraic group  $\mathcal{G}_i$  over  $\mathbb{F}_{q^{r_i}}$  realized inside  $GL(W_i)$  such that  $G^{(i)} = Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})$  and  $Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}})) \subset Z(G^{(i)})$ . In particular, the natural map*

$$G_1/G_2 \longrightarrow \prod_{1 \leq i \leq m} G^{(i)}/Z(G^{(i)}) \simeq \prod_{1 \leq i \leq m} \mathcal{G}_i(\mathbb{F}_{q^{r_i}})/Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}}))$$

*is isomorphic where each component of the right hand side is a simple Chevalley group (cf. [19]). Further  $G_1 \hookrightarrow \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})$  with respect to the decomposition  $V = \bigoplus_{1 \leq i \leq m} W_i$ , and there exist a constant  $C_1(n)$  depending only on  $n$  so that*

$$|G_1| \geq C_1(n) \prod_{1 \leq i \leq m} |\mathcal{G}_i(\mathbb{F}_{q^{r_i}})|.$$

*Proof.* The first part of (1) follows from Proposition 5.9-(2). Since  $(G_1, W_i)$  is (absolutely) irreducible, by Clifford's theorem,  $W_i|_{G_2}$  decomposes into isotypical representations of 1-dimensional representations. Hence  $\pi_i(G_2)$  are scalar matrices. Hence it clearly commutes with  $G^{(i)}$ . The latter claim is clear from the injectivity of  $G_1 \hookrightarrow \prod_{i=1}^m G^{(i)}$ .

We now prove the second claim. Let  $\Gamma_i^0$  be the group generated by all elements of  $p$ -th power order in  $G^{(i)}$ . Then by Theorem B of [12] (see also step 1 in the proof of Proposition A.7 of [13]),  $\Gamma_i^0/Z(\Gamma_i^0)$  is a simple Chevalley group. Since  $\Gamma_i^0$  is a normal subgroup of  $G^{(i)}$ , so is  $Z(G^{(i)}) \cdot \Gamma_i^0/Z(G^{(i)})$  in  $G^{(i)}/Z(G^{(i)})$ . However  $G^{(i)}/Z(G^{(i)})$  is by construction (see Theorem 5.1-(b)), a simple group. Then one has  $Z(G^{(i)}) \cdot \Gamma_i^0/Z(G^{(i)}) = G^{(i)}/Z(G^{(i)})$ . Hence  $Z(G^{(i)})\Gamma_i^0 = G^{(i)}$ . The surjective map  $\Gamma_i^0 \longrightarrow G^{(i)}/Z(G^{(i)})$  induces an isomorphism  $\Gamma_i^0/Z(\Gamma_i^0) \xrightarrow{\sim} G^{(i)}/Z(G^{(i)})$ . If an element  $g \in Z(\Gamma_i^0)$  does not belong to  $Z(G^{(i)})$ , then the previous isomorphism never be

isomorphic, hence it gives a contradiction. Hence one has  $Z(\Gamma_i^0) \subset Z(G^{(i)}) \subset \mathbb{F}_{q^{r_i}}^\times \text{id}_{W_i}$ . This means that the image of  $\Gamma_i^0$  under the projective map  $\text{GL}(W_i) \longrightarrow \text{PGL}(W_i)$  is  $\Gamma_i^0/Z(\Gamma_i^0)$  and it is a simple Chevalley group. Then the claim follows by looking any simple Chevalley group which appears in this way (see [19]). Therefore there exists a simple and simply connected algebraic group  $\mathcal{G}_i$  over  $\mathbb{F}_{q^{r_i}}$  such that  $G^{(i)} = Z(G^{(i)})\Gamma_i^0 = Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})$ .

To prove the last claim, let us consider the following commutative diagram

$$\begin{array}{ccc} G_1 & \longrightarrow & \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}}) \\ \pi_1 \downarrow & & \pi_2 \downarrow \\ G_1/G_2 & \xrightarrow{\sim} & \prod_{1 \leq i \leq m} \mathcal{G}_i(\mathbb{F}_{q^{r_i}})/Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}})). \end{array}$$

where the top arrow is an injective map which is defined by the decomposition  $V = \bigoplus_{1 \leq i \leq m} W_i$ . Here  $\pi_1$  and  $\pi_2$  stand for natural projections. Then one has

$$|G_1| \geq C_1(n) \prod_{1 \leq i \leq m} |\mathcal{G}_i(\mathbb{F}_{q^{r_i}})|, \quad C_1(n) := \frac{1}{\prod_{1 \leq i \leq m} |Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}}))|}.$$

Since  $\mathcal{G}_i$  is simple and simply connected algebraic group, the cardinality of  $Z(\mathcal{G}_i(\mathbb{F}_{q^{r_i}}))$  is depending only on the rank of  $\mathcal{G}_i$ , hence on  $n$ . This completes the proof.  $\square$

For any subgroup  $D \subset \text{GL}_n(\mathbb{F}_q)$  and each  $g \in D$ , we define by  $M_D(g)$  the number of elements of  $D$  which have the same characteristic polynomial as  $g$  and put  $M_D = \max_{g \in D} \{M_D(g)\}$ . Note that for any subgroup  $A$  of the center  $\mathbb{F}_q^\times I_n$ ,  $M_{A \cdot D} = M_D$  and  $M_{D_1} \leq M_{D_2}$  for subgroups  $D_1 \subset D_2 \subset \text{GL}_n(\mathbb{F}_q)$ . This simple observations will be used in the proof of Theorem 5.13 below.

**Lemma 5.11.** *For each  $i$  ( $1 \leq i \leq m$ ), let  $D_i$  be a subgroup of  $\text{GL}_{n_i}(\mathbb{F}_{q^{r_i}})$ . We identify the product  $D := \prod_{1 \leq i \leq m} D_i$  with the Levi subgroup of the parabolic subgroup  $P_{(n_1, \dots, n_m)}$  in  $\text{GL}_n(\overline{\mathbb{F}}_q)$  with respect to the partition  $n = n_1 + \dots + n_m$ . Then there exists a constant  $C_2(n)$  depending only on  $n$  so that*

$$M_D \leq C_2(n) \prod_{1 \leq i \leq m} M_{D_i}.$$

*Proof.* We will give a very rough estimation for  $C_2(n)$ . For any  $g = (g_1, \dots, g_m)$ ,  $f_g(T) = \prod_{1 \leq i \leq m} f_{g_i}(T)$ . We denote the eigenvalues by  $\alpha_1^{(1)}, \dots, \alpha_{n_1}^{(1)}, \alpha_1^{(2)}, \dots, \alpha_{n_2}^{(2)}, \dots, \alpha_1^{(m)}, \dots, \alpha_{n_m}^{(m)}$  which

are not necessarily different from each other. Then the number of all permutations which preserve the type  $(n_1, \dots, n_m)$  is  $\frac{n!}{n_1! \cdots n_m!}$ . We may take this as  $C_2(n)$ .  $\square$

**Proposition 5.12.** (*Proposition 3.1 of [18] or Lemma 3.5 of [22]*) *For any connected algebraic group  $G$  over  $\mathbb{F}_q$ , we have*

$$(\sqrt{q} - 1)^{2\dim G} \leq |G(\mathbb{F}_q)| \leq (\sqrt{q} + 1)^{2\dim G}.$$

*For connected linear algebraic groups, one has a stronger estimate*

$$(q - 1)^{\dim G} \leq |G(\mathbb{F}_q)| \leq (q + 1)^{\dim G}.$$

Let  $G$  be a simple and simply connected algebraic group over a finite field  $\mathbb{F}_q$ . We follow [6] for the following proposition.

**Proposition 5.13.** *Let  $l = \text{rank}(G)$ , and let  $A$  be a semisimple element in  $G$  and  $C(A)$  be the centralizer of  $A$  in  $G(\mathbb{F}_q)$ , and  $d = \dim C(A)$ . Then*

$$\frac{q^d}{(q+1)^d} \frac{|G(\mathbb{F}_q)|}{q^l} \leq M_G(A) \leq \frac{q^d}{(q-1)^d} \frac{|G(\mathbb{F}_q)|}{q^l}.$$

*Proof.* Let  $\Delta_G(A)$  be the set of  $g \in G$  which has the same characteristic polynomial as  $A$  so that  $M_G(A) = |\Delta_G(A)|$ . Suppose  $g \in \Delta_G(A)$ . Then  $g = g_s g_u$  with  $g_s$  semisimple,  $g_u$  unipotent. Then  $\det(1 - Tg_s) = \det(1 - TA)$ .

Since  $g_s$  and  $A$  are conjugate in  $G(\overline{\mathbb{F}_q})$ , they are conjugate in  $G(\mathbb{F}_q)$  [29]. Over  $\overline{\mathbb{F}_q}$ , the algebraic group  $C(A)$  is the centralizer in  $G(\overline{\mathbb{F}_p})$  of  $A$ . Since  $A$  is semisimple and  $G$  is simply connected,  $C(A)$  is a connected reductive group [28]. Since  $C(A)$  contains any maximal torus of  $G(\mathbb{F}_p)$  and  $C(A) \subset G(\mathbb{F}_p)$ ,  $\text{rank}(C(A)) = l$ . By Steinberg [28],

$$\#\{\text{unipotent elements in } C(A)(\mathbb{F}_q)\} = q^{d-l}.$$

Therefore,

$$\begin{aligned} M_G(A) &= \#\{\text{pairs } (g_s, g_u) \mid g_s \text{ is } G(\mathbb{F}_p)\text{-conjugate to } A \text{ and } g_u \in (C(g_s))_u(\mathbb{F}_q)\} \\ &= q^{d-l} \#\{g_s \text{ which is } G(\mathbb{F}_q)\text{-conjugate to } A\} = q^{d-l} \frac{\#G(\mathbb{F}_q)}{\#C(A)(\mathbb{F}_q)}. \end{aligned}$$

Since  $C(A)$  is connected, by Proposition 5.12,  $(q - 1)^d \leq \#C(A)(\mathbb{F}_p) \leq (q + 1)^d$ . Hence our assertion follows.  $\square$

Here  $M_G(A) \leq K \frac{|G(\mathbb{F}_q)|}{q^l}$  for a constant  $K$  depending only on  $\dim G$ .

**Theorem 5.14.** *Let  $G$  be a semisimple subgroup of  $GL_n(\mathbb{F}_q)$ . Assume that  $p > C(n)$ . If  $G$  satisfies the property  $C(\eta, N)$  for  $0 < \eta < \frac{1}{C(n)}$ , then either  $|G|$  or  $q$  is bounded by a constant depending only on  $n$ . Hence  $|G| \leq B = B(\eta, N, n)$ .*

*Proof.* Take a series of normal subgroups  $G \supset G_1 \supset G_2 \supset G_3$  in Theorem 5.1. By Corollary 5.5,  $G_3 = \{1\}$ . If  $G_1 = G_2$ , then the claim follows from Proposition 5.8.

Henceforth we assume  $G_1 \neq G_2$ . Then by Proposition 5.10, there exists a injective map

$$G_1 \longrightarrow \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}}), \quad \mathcal{G}_i(\mathbb{F}_{q^{r_i}}) \subset GL_{n_i}(\mathbb{F}_{q^{r_i}}), \quad 1 \leq i \leq m.$$

Then one has

$$M_{G_1} \leq M_D, \quad D := \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}}).$$

Since  $G$  satisfies  $C(\eta, N)$ , by Lemma 5.7,  $G_1$  satisfies  $C(C(n)\eta, N)$ . This means that

$$(1 - C(n)\eta)|G_1| \leq |H|.$$

Then applying Proposition 5.13 to  $D = \prod_{1 \leq i \leq m} Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})$ , and by Lemma 5.11, one has

$$\begin{aligned} (1 - C(n)\eta)C_1(n) \prod_{1 \leq i \leq m} |\mathcal{G}_i(\mathbb{F}_{q^{r_i}})| &\leq (1 - C(n)\eta)|G_1| \leq |H| \leq NM_{G_1} \\ &\leq NM_D \leq NC_2(n) \prod_{1 \leq i \leq m} M_{Z(G^{(i)})\mathcal{G}_i(\mathbb{F}_{q^{r_i}})} = NC_2(n) \prod_{1 \leq i \leq m} M_{\mathcal{G}_i(\mathbb{F}_{q^{r_i}})} \\ &\leq NK(n)C_2(n) \prod_{1 \leq i \leq m} \frac{|\mathcal{G}_i(\mathbb{F}_{q^{r_i}})|}{q^{r_i l_i}} \end{aligned}$$

with a constant  $K(n)$ , where  $l_i = \text{rank } \mathcal{G}_i$ . This gives us the bound

$$q \leq \prod_{1 \leq i \leq m} q^{r_i l_i} \leq \frac{NK(n)C_2(n)}{(1 - C(n)\eta)C_1(n)}.$$

Hence the claim follows.  $\square$

**Corollary 5.15.** *Let  $S$  be an infinite set of rational primes. Suppose for each prime  $\ell \in S$ , the image of a mod  $\ell$  semisimple Galois representation  $\rho_\ell : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{F}_\ell)$  satisfies  $C(\eta, N)$  for  $0 < \eta < \frac{1}{C(n)}$ . Then there exists a constant  $A = A(\eta, N, n)$  such that  $|\text{Im } \rho_\ell| \leq A$ .*

6. MOD  $\ell$  REPRESENTATIONS

We assume the following existence of mod  $\ell$  representations.

**Conjecture 6.1.** *Let  $\pi$  be as in Section 2. Let  $\ell$  be an odd prime which is coprime to  $N$ . Then for each finite place  $\lambda$  of  $\mathbb{Q}_\pi$  with the residue field  $\mathbb{F}_\lambda$ , there exists a continuous semi-simple representation*

$$\rho_\lambda : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{F}}_\lambda)$$

*which is unramified outside of  $\ell N$ , so that*

$$\det(I_n - \rho_\lambda(\text{Frob}_p)T) \equiv H_p(T) \pmod{\lambda},$$

*for any  $p \nmid \ell N$ , where  $H_p(T) = 1 - a_1(p)T + a_2(p)T^2 + \cdots + (-1)^n a_n(p)T^n$ .*

## 7. ARTIN REPRESENTATIONS ASSOCIATED TO CUSPIDAL REPRESENTATIONS

In this section we give a proof of the main theorem (Theorem 1.1). Let  $\pi$  be as in Section 2. We denote by  $S_\pi$  the set of rational primes consisting of primes  $p$  so that  $\pi_p$  is ramified. Let  $K$  be a Galois closure of  $\mathbb{Q}_\pi$ . By assumptions on  $\pi$ , this is a finite extension of  $\mathbb{Q}$ . Let  $P_K$  be the set prime numbers  $\ell$  which splits completely in  $K$ . For each  $\ell \in P_K$ , choose a finite place  $\lambda_\ell$  of  $K$  dividing  $\ell$ . By Conjecture 6.1, there exists a continuous semi-simple representation

$$\rho_\ell : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{F}}_\ell)$$

which is unramified outside  $S_\pi \cup \{\ell\}$ , and

$$\det(I_n - \rho_\ell(\text{Frob}_p)T) \equiv H_p(T) \pmod{\lambda_\ell}.$$

By Lemma 6.13 of [8], we may assume that the image of  $\rho_\ell$  takes the values in  $GL_n(\mathbb{F}_\ell)$ . Let  $G_\ell := \text{Im } \rho_\ell$ .

**Lemma 7.1.** *For any  $\eta$ ,  $0 < \eta < 1$ , there exists a constant  $M$  such that  $G_\ell$  satisfies  $C(\eta, M)$  for every  $\ell \in P_K$ .*

*Proof.* By Proposition 4.1, if we let  $\mathcal{M} := \{H_p(T) \mid p \notin X_\eta\}$ , then  $\mathcal{M}$  is a finite set. Let  $M := |\mathcal{M}|$  which will be a desired constant as below. Let us consider the subset of  $G_\ell$  defined by

$$H_\ell := \{g \in G_\ell \mid g \stackrel{G_\ell}{\sim} \rho_\ell(\text{Frob}_p) \text{ for some } p \notin X_\eta\}.$$

By Chebotarev density theorem, one has

$$1 = \frac{|H_\ell|}{|G_\ell|} + \text{den}(X_\eta) \leq \frac{|H_\ell|}{|G_\ell|} + \text{den.sup}(X_\eta) \leq \frac{|H_\ell|}{|G_\ell|} + \eta,$$

giving  $(1 - \eta)|G_\ell| \leq |H_\ell|$ .

The eigen polynomial of each element of  $H_\ell$  is the reduction of some element of  $\mathcal{M}$ . Therefore one has

$$|\{\det(I_n - hT) \mid h \in H_\ell\}| \leq M.$$

□

By Lemma 7.1 together with Corollary 5.15, there exists a constant  $A$  such that  $|G_\ell| \leq A$  for any  $\ell \in P_K$ . Let  $Y$  be the set of polynomials  $\prod_{i=1}^n (1 - \alpha_i T)$ , where  $\alpha_i$ 's are roots of unity of order less than  $A$ . If  $p \notin S_\pi$ , for all  $\ell \in P_K$  with  $\ell \neq p$ , there exists  $R(T) \in Y$  such that

$$H_p(T) \equiv R(T) \pmod{\lambda_\ell}.$$

Since  $Y$  is a finite set, there exists  $R(T) \in Y$  such that for infinitely many  $\ell \in P_K$ ,

$$H_p(T) = R(T).$$

Let  $P'_K$  be the set of  $\ell \in P_K$  such that  $\ell > A$  and for  $R, S \in Y$  with  $R \neq S$ ,  $R \not\equiv S \pmod{\lambda_\ell}$ . Then it is easy to see that  $P'_K$  is infinite. For each  $\ell \in P'_K$ ,  $\ell$  does not divide  $|G_\ell|$ , since  $\ell > A \geq |G_\ell|$ . Let  $\pi_\ell : GL_n(\mathcal{O}_{\lambda_\ell}) \rightarrow GL_n(\mathbb{F}_\ell)$  be the reduction modulo  $\lambda$  whose kernel  $P$  is a pro  $\ell$ -group. By applying Schur-Zassenhaus' theorem (cf. [9], page 829) to the projection  $\pi_\ell$  and  $G_\ell \subset GL_n(\mathbb{F}_\ell)$ , there exists a subgroup  $H \subset GL_n(\mathcal{O}_{\lambda_\ell})$  such that  $\pi$  induces an isomorphism  $H \xrightarrow{\sim} G_\ell = \text{Im } \rho_\ell$ . Hence we have a lift  $\rho'_\ell : G_\mathbb{Q} \rightarrow GL_n(\mathcal{O}_{\lambda_\ell})$  of  $\rho_\ell$ . Since the coefficient of  $\rho'_\ell$  is of characteristic zero and its image is finite, for  $p \nmid N\ell$ , one has  $\det(I_n - \rho'_\ell(\text{Frob}_p)T) \in Y$ . On the other hand, we have

$$\det(I_n - \rho'_\ell(\text{Frob}_p)T) \equiv H_p(T) \pmod{\lambda_\ell}.$$

Since  $\ell \in P'_K$ , the above congruence relation implies the equality

$$\det(I_n - \rho'_\ell(\text{Frob}_p)T) = H_p(T).$$

for all  $p \nmid N\ell$ . Now we replace  $\ell$  with another prime  $\ell' \in P'_K$ . Then one has  $\rho'_{\ell'} : G_\mathbb{Q} \rightarrow GL_n(\mathcal{O}_{\lambda_{\ell'}})$  such that

$$\det(I_n - \rho'_{\ell'}(\text{Frob}_p)T) = \det(I_n - \rho'_\ell(\text{Frob}_p)T)$$

for all  $p \nmid N\ell\ell'$ . By Chebotarev density theorem, one has  $\rho'_{\ell'} \sim \rho'_\ell$  and this means that  $\rho'_\ell$  is unramified at  $\ell$ . Hence we have the desired representation

$$\rho_\pi := \rho'_\ell : G_{\mathbb{Q}} \longrightarrow GL_n(\mathcal{O}_{\lambda_\ell}) \hookrightarrow GL_n(\mathbb{C}),$$

where the second map comes from a fixed embedding  $\mathcal{O}_{\lambda_\ell} \hookrightarrow \mathbb{C}$ . This representation is independent of any choice of such a embedding by Chebotarev density theorem.

Finally, if  $c$  is the complex conjugate, it is clear that  $\rho_\pi(c) \stackrel{GL_n(\mathbb{C})}{\sim} \text{diag}(\epsilon_1, \dots, \epsilon_n)$ ,  $\epsilon_i \in \{\pm 1\}$ . Hence the Langlands' parameter for  $\pi_\infty$  is

$$\phi : W_{\mathbb{R}} \longrightarrow GL_n(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \text{diag}(\epsilon_1, \dots, \epsilon_n).$$

$$\text{Hence } \pi_\infty \simeq \pi(\epsilon'_1, \dots, \epsilon'_n) \text{ where } \epsilon'_i = \begin{cases} 1, & \text{if } \epsilon_i = 1 \\ \text{sgn}, & \text{if } \epsilon_i = -1 \end{cases}.$$

**Corollary 7.2.** *Let  $\pi$  be a cuspidal representation of  $GL_n/\mathbb{Q}$  as in Section 2. Then  $\pi_p$  is tempered for all  $p$ .*

*Proof.* By Theorem 1.1, there exists the Artin representation  $\rho_F : G_{\mathbb{Q}} \longrightarrow GL_n(\mathbb{C})$  such that  $L(s, \rho_p) = L(s, \pi_p)$  for almost all  $p$ . By Proposition A.1 of [21],  $L(s, \rho_p) = L(s, \pi_p)$  for all  $p$ . Hence  $\pi_p$  is tempered for all  $p$ .  $\square$

## 8. NON-HOLOMORPHIC SIEGEL MODULAR FORMS AND HOLOMORPHIC SIEGEL MODULAR FORMS VIA THE CONGRUENCE METHOD

In this section we follow the notation of [16]. Let us first recall the existence of a Galois representation for any holomorphic Siegel modular form of weight  $(k_1, k_2)$  for  $GSp_4$ . Thanks to the works of [20] and [31] with the classification of CAP forms ([23], [27], [26]) and endoscopic representations for  $GSp_4$  ([24]), we can associate a Galois representation to  $F$ .

**Theorem 8.1.** *For any prime  $\ell$ , there exists a number field  $E$  including  $\mathbb{Q}_F$ , such that for each rational prime  $\ell$  and a finite place  $\lambda|\ell$  of  $\mathbb{Q}_F$ , there exists a continuous representation  $\rho_{F,\ell} : G_{\mathbb{Q}} \longrightarrow GL_4(E_\lambda)$  which is unramified outside of  $\ell N$  so that*

$$\det(1_4 - \rho_{F,\ell}(\text{Frob}_p)p^{-s})^{-1} = L_p(s, F) = L_p(s - \frac{k_1+k_2-3}{2}, \pi_F)$$

for any  $p \nmid \ell N$ . Furthermore, if  $k_1 \geq k_2 \geq 3$  and  $\pi_F$  is neither endoscopic nor CAP, then the image of  $\rho_{F,\ell}$  can be taken in  $GSp_4(E_\lambda)$ .

Let us denote by  $\overline{\rho}_{F,\ell} : G_{\mathbb{Q}} \longrightarrow GL_4(\mathbb{F})$  the reduction modulo  $\lambda(\lambda|\ell)$  where  $\mathbb{F}$  is the residue field of  $\lambda$ .

Let  $f$  be an elliptic newform of weight one which is neither of dihedral nor of tetrahedral type. Then this gives rise to a unique Artin representation  $\rho_f : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{C})$ . Since the image is finite, we can take a finite extension  $K$  of  $\mathbb{Q}$  so that  $Im(\rho_f) \subset GL_2(\mathcal{O})$  where  $\mathcal{O}$  is the ring of integers of  $K$ . Then taking the reduction modulo a prime ideal above a rational prime  $\ell$ , we obtain a mod  $\ell$  representation  $\overline{\rho}_{f,\ell} : G_{\mathbb{Q}} \longrightarrow GL_2(\mathbb{F})$ .

By Theorem 10.1 of [16], there exists a real analytic Siegel modular form  $F$  of weight  $(2, 1)$  with eigenvalues  $-\frac{5}{12}$  (resp. 0) for  $\Delta_1$  (resp.  $\Delta_2$ ) (see [16] for  $\Delta_i$ ) such that  $\pi_F \sim Sym^3(\pi_f)$ . By using this, we obtain a mod  $\ell$  representation  $\pi_{F,\ell} : G_{\mathbb{Q}} \longrightarrow GL_4(\mathbb{F})$  for  $F$ .

On the other hand, by multiplying Hasse invariant of weight  $\ell - 1$ , we obtain an eigenform  $g$  of weight  $1 + a(p - 1)$  for any positive integer  $a$  such that  $g$  is congruent to  $f$  modulo  $\ell$  hence  $\overline{\rho}_{g,\ell} \simeq \overline{\rho}_{f,\ell}$ . By using symmetric cube lift and generic transfer from  $GSp_4$  to  $GL_4$ , one can show the existence of a holomorphic Siegel cusp form  $G$  of weight  $(k_1, k_2) = (2a(\ell - 1) + 2, a(\ell - 1) + 1)$  such that  $\rho_{G,\ell} = Sym^3(\rho_{g,\ell})$ . From this one concludes that there exist a non-holomorphic Siegel modular form  $F$  of weight  $(2, 1)$  and a holomorphic Siegel modular form  $G$  of weight  $(2a(\ell - 1) + 2, a(\ell - 1) + 1)$  such that

$$\overline{\rho}_{F,\ell} \simeq \overline{\rho}_{G,\ell}.$$

We denote by this property  $F \equiv G \pmod{\ell}$  provided if the existence of  $\ell$ -adic and mod  $\ell$  representations of  $F$  and  $G$  are guaranteed.

We can also construct such  $F$  and  $G$  by using endoscopic lift from a pair  $(f_1, f_2), \pi_{f_1} \not\sim \pi_{f_2}$  of elliptic newform of weight one whose central characters are same as follows. By using theta lift (cf [24]) and Section 5 of [16], there exists a real analytic Siegel modular form of weight  $(2, 1)$  as above such that  $\rho_{F,\ell} \simeq \rho_{f_1} \oplus \rho_{f_2}$ . By multiplying Hasse invariant again, one has a pair of elliptic modular forms  $(g_1, g_2), \pi_{g_1} \not\sim \pi_{g_2}$  of elliptic newform  $g_1$  (resp.  $g_2$ ) of weight  $r_1 = 1 + a(\ell - 1)$  (resp.  $r_2 = 1 + b(\ell - 1)$ ) with the same central character. Then by using theta lift (cf [24]), one can construct a holomorphic Siegel cusp form  $G$  of weight  $(k_1, k_2) = (\frac{(\ell-1)(a+b)}{2} + 1, \frac{(\ell-1)(a-b)}{2} + 2)$  such that  $\rho_{F,\ell} \simeq \rho_{g_1,\ell} \oplus \rho_{g_2,\ell}$ . Taking reduction modulo  $\ell$ , one concludes  $F \equiv G \pmod{\ell}$ .

For such  $F$  (and  $G$ ), the mod  $\ell$  representation  $\overline{\rho}_{F,\ell}$  has a remarkable property that  $\overline{\rho}_{F,\ell}$  is unramified at  $\ell$ . In case elliptic newform, this property characterizes a weight  $\ell$  form so that it comes from a weight one form by multiplying Hasse invariant of weight  $\ell - 1$ . This principle is



discussed in Proposition 2.7 of [10] which plays an important role for proving Serre conjecture. So this gives rise to the following natural question:

**Question 8.2.** *Let  $G$  be a holomorphic Siegel cusp form of weight  $(k_1, k_2)$  so that  $k_1 - 1$  and  $k_2 - 2$  are both divided by  $\ell - 1$ , where  $\ell$  is a rational prime. Assume that  $\bar{\rho}_{G, \ell}$  is unramified at  $\ell$ . Can one associate a non-holomorphic Siegel cusp form  $F$  of weight  $(2, 1)$  with a mod  $\ell$  representation such that  $F \equiv G \pmod{\ell}$ ?*

## 9. SUPPLEMENT TO OUR PAPER [16]

In [16], we used Arthur's conjectural result on the correspondence between cuspidal representations of  $GSp_4$  and  $GL_4$  [2]. It depends on the stabilization of the trace formula, which is not proved yet. In this section, we explain how to get around this by using the transfer from  $Sp_4$  to  $GL_5$  in [3]. The result depends on the twisted fundamental lemma which may have been resolved by now.

Let  $\pi = \pi_F$  be the cuspidal representation of  $GSp_4$  attached to the Siegel cusp form  $F$  of weight  $(2, 1)$ . We showed in [16] that  $\pi_F$  is not a CAP representation. Let  $\pi'$  be one of components of  $\pi|_{Sp_4(\mathbb{A})}$ . Then it is a cuspidal representation of  $Sp_4/\mathbb{Q}$ . By [3],  $\pi'$  corresponds to an automorphic representation  $\Pi_5$  of  $GL_5$ . Since  $\pi'$  is not a CAP representation,  $\Pi_5$  is either cuspidal or an isobaric representation.

By using the descent construction [11], we can find a globally generic cuspidal representation  $\tau'$  of  $Sp_4$  which is in the same  $L$ -packet as  $\pi'$ . Now let  $\tau$  be a globally generic cuspidal representation of  $GSp_4$  such that  $\tau'$  occurs in the restriction  $\tau|_{Sp_4(\mathbb{A})}$ . By [4], we have a functorial lift  $\Pi$  of  $\tau$  as an automorphic representation of  $GL_4$ . This  $\Pi$  is the transfer of  $\pi$ . We can see easily that  $\wedge^2(\Pi) = \Pi_5 \otimes \omega_\pi \boxplus \omega_\pi$ , i.e.,  $\Pi_5$  is the transfer of  $\pi$  to  $GL_5$  corresponding to the  $L$ -group homomorphism  $GSp_4(\mathbb{C}) \rightarrow GL_5(\mathbb{C})$ . Hence we do not need the exterior square lift of  $\Pi$  in [15] in order to obtain  $\Pi_5$ .

## 10. ARTIN REPRESENTATIONS ATTACHED TO CUSP FORMS ON IMAGINARY QUADRATIC FIELDS

In this section, as a supplement to our paper [16], we use the idea of [7] to construct a non-holomorphic Siegel cusp form of weight  $(2, 1)$  attached to Maass forms for  $GL_2/\mathbb{Q}$  and cuspidal representations of  $GL_2$  over imaginary quadratic fields.

Let  $K = \mathbb{Q}[\sqrt{-D}]$  be an imaginary quadratic field. Let  $Gal(K/\mathbb{Q}) = \{1, \theta\}$ , and  $\omega_{K/\mathbb{Q}}$  be the quadratic character attached to  $K/\mathbb{Q}$  i.e.,  $\omega_{K/\mathbb{Q}}(p) = (\frac{-D}{p})$ .

Let  $\mathbf{G} = R_{K/\mathbb{Q}}GL_2$  be the quasi-split group obtained by the restriction of scalars. Then  ${}^L\mathbf{G} = (GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \rtimes Gal(K/\mathbb{Q})$ , and  $\mathbf{G}(\mathbb{A}) = GL_2(\mathbb{A}_K)$ . Let  $\pi = \pi_\infty \otimes \otimes'_p \pi_p$  be a cuspidal representation of  $\mathbf{G}(\mathbb{A})$ . Here  $\pi_\infty$  is a unitary representation of  $GL_2(\mathbb{C})$ . If  $p$  splits in  $K$  into  $(v_1, v_2)$ , then  $\pi_p = \pi_{v_1} \otimes \pi_{v_2}$ . We make the following assumption on  $\pi$ .

**Assumption 10.1.**  $\omega_\pi$  factors through  $N_{K/\mathbb{Q}}$ , i.e.,  $\omega_\pi = \omega \circ N_{K/\mathbb{Q}}$  with a grössencharacter  $\omega$ .

The automorphic induction corresponds to the  $L$ -group homomorphism

$$I_K^{\mathbb{Q}} : {}^L\mathbf{G} \longrightarrow GL(\mathbb{C}^2 \oplus \mathbb{C}^2) \simeq GL_4(\mathbb{C}), \quad I_K^{\mathbb{Q}}(g, g'; 1)(x \oplus y) = g(x) \oplus g'(y), \quad I_K^{\mathbb{Q}}(1, 1; \theta)(x \oplus y) = y \oplus x.$$

Let  $I_K^{\mathbb{Q}}\pi$  be the automorphic induction. It is automorphic representation of  $GL_4/\mathbb{Q}$ , and it is not cuspidal if and only if  $\pi \simeq \pi \circ \theta$ . In that case,  $\pi$  is a base change of a cuspidal representation  $\pi_0$  of  $GL_2/\mathbb{Q}$ , and  $I_K^{\mathbb{Q}}\pi = \pi_0 \boxplus (\pi_0 \otimes \omega_{K/\mathbb{Q}})$ .

The Asai lift corresponds to the  $L$ -group homomorphism

$$As : {}^L\mathbf{G} \longrightarrow GL(\mathbb{C}^2 \otimes \mathbb{C}^2) \simeq GL_4(\mathbb{C}), \quad As(g, g'; 1)(x \otimes y) = g(x) \otimes g'(y), \quad As(1, 1; \theta)(x \otimes y) = y \otimes x.$$

If  $\rho : G_K \longrightarrow GL_2(\mathbb{C})$ , we have [14]

$$\wedge^2(\text{Ind}_K^{\mathbb{Q}}(\rho)) = (As(\rho) \otimes \omega_{K/\mathbb{Q}}) \oplus \text{Ind}_K^{\mathbb{Q}}(\det \rho).$$

Hence if  $\rho$  corresponds to  $\pi$ ,  $\det(\rho)$  corresponds to  $\omega_\pi$ . If  $\pi$  satisfies Assumption 10.1,  $I_K^{\mathbb{Q}}\omega_\pi = \omega \oplus \omega\omega_{K/\mathbb{Q}}$ . Hence we can  $L(s, \wedge^2(I_K^{\mathbb{Q}}) \otimes \chi^{-1})$  with  $\chi = \omega$  or  $\omega\omega_{K/\mathbb{Q}}$ , has a pole at  $s = 1$ , and  $I_K^{\mathbb{Q}}\pi$  descends to a cuspidal representation of  $GS_{p_4}/\mathbb{Q}$  with the central character  $\chi$  (cf. [5]).

Let  $\pi_\infty = \pi(1, 1)$ . Then the Langlands' parameter of  $\pi_\infty$  is

$$\phi : W_{\mathbb{C}} = \mathbb{C}^\times \longrightarrow (GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \rtimes Gal(K/\mathbb{Q}), \quad \phi(z) = (I, I; \theta).$$

So the Langlands' parameter of  $I_K^{\mathbb{Q}}(\pi_\infty)$  is

$$\phi : W_{\mathbb{R}} \longrightarrow GL_4(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

$$\text{Here if } P = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix},$$

$$P^{-1} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} P = \text{diag}(1, -1, -1, 1), \quad {}^t P \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} P = -\frac{1}{2} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

So  $\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$  is conjugate to  $\text{diag}(1, -1, -1, 1)$  in  $GSp_4(\mathbb{C})$ , and up to conjugacy, we have

$$\phi : W_{\mathbb{R}} \longrightarrow GSp_4(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \text{diag}(1, -1, -1, 1).$$

Then  $\phi$  is the Langlands' parameter for  $\text{Ind}_B^{GSp_4} 1 \otimes \text{sgn} \otimes \text{sgn}$ , and as in [16], we can show that there exists a Siegel cusp form  $F$  of weight  $(2, 1)$  corresponding to  $I_K^{\mathbb{Q}} \pi$ . We have proved

**Theorem 10.1.** *Let  $\pi$  be a cuspidal representation of  $GL_2/K$ ,  $K = \mathbb{Q}[\sqrt{-D}]$  which satisfies Assumption 10.1, and  $\pi_{\infty} = \pi(1, 1)$ . Then there exists a non-holomorphic Siegel cusp form  $F$  of weight  $(2, 1)$  such that  $L(s, \pi_F) = L(s, \pi)$ .*

**10.1. Artin representation attached to Maass forms.** Let  $\pi$  be a cuspidal representation of  $GL_2/\mathbb{Q}$  such that  $\pi_{\infty} = \pi(1, 1)$ , i.e., Maass cusp form. The Langlands parameter of  $\pi_{\infty}$  is

$$\phi : W_{\mathbb{R}} \longrightarrow GL_2(\mathbb{C}), \quad \phi(z) = I_2, \quad \phi(j) = I_2.$$

Let  $BC(\pi)$  be the base change to  $K = \mathbb{Q}[\sqrt{-D}]$ , and consider

$$\Pi = I_K^{\mathbb{Q}}(BC(\pi)) = \pi \boxplus (\pi \otimes \omega_{K/\mathbb{Q}}).$$

Then  $\Pi$  descends to a generic cuspidal representation  $\tau$  of  $GSp_4$  (cf. [5]). The Langlands parameter of  $\Pi_{\infty}$  is

$$\phi : W_{\mathbb{R}} \longrightarrow GL_4(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \text{diag}(1, 1, -1, -1).$$

Then we can show easily that for  $s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ , which is the Weyl group element corresponding to the long simple root,

$$s_2^{-1} \text{diag}(1, 1, -1, -1) s_2 = \text{diag}(1, -1, -1, 1).$$

Hence  $\text{diag}(1, 1, -1, -1)$  and  $\text{diag}(1, -1, -1, 1)$  are conjugate in  $GS p_4(\mathbb{C})$ , and up to conjugacy, we have

$$\phi : W_{\mathbb{R}} \longrightarrow GS p_4(\mathbb{C}), \quad \phi(z) = Id, \quad \phi(j) = \text{diag}(1, -1, -1, 1).$$

Since  $\phi$  is the Langlands' parameter for  $\text{Ind}_B^{GS p_4} 1 \otimes sgn \otimes sgn$ , as in [16], we can show that there exists a Siegel cusp form  $F$  of weight  $(2, 1)$  corresponding to  $I_K^{\mathbb{Q}}(BC(\pi))$ . We have proved

**Theorem 10.2.** *Let  $\pi$  be a cuspidal representation of  $GL_2/\mathbb{Q}$  such that  $\pi_{\infty} = \pi(1, 1)$ . Then there exists a non-holomorphic Siegel cusp form  $F$  of weight  $(2, 1)$  such that  $L(s, \pi_F) = L(s, \pi)L(s, \pi \otimes \omega_{K/\mathbb{Q}})$ .*

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